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Scaling-law for the maximal Lyapunov exponent

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Abstract. We study the scaling law for $\varepsilon \rightarrow 0$ of the maximal Lyapunov exponent for coupled chaotic map lattices and for products of random Jacobi matrices. To this purpose we develop approximate analytical treatments of the random matrix problem inspired by the theory of directed polymers in a random medium: a type of mean-field method and a tree approximation which introduces correlations. The theoretical results suggest a leading $|\log \varepsilon|^{-1}$ increase in the maximal Lyapunov exponent near $\varepsilon = 0$, which is confirmed by numerical simulations, also for coupled map lattices. A dynamical mechanism responsible for this behaviour is investigated for a 2×2 random matrix model. The theory also predicts a phase transition at a critical value of the coupling ε_c , which is not observed in numerical simulations and might be an artifact of the approximation.

1. Introduction

The appearance of defect-mediated dynamics, the spatio-temporal intermittency typical of certain fluid regimes, the spiralling waves observed in chemical reaction-diffusion processes, and the self-regulatory behaviour of population dynamics are all different manifestations of the complex features of spatially extended dynamical systems. Considerable effort has been devoted in the last years to include common properties and peculiarities of these systems in a general framework. The main ingredients, particularly fit for a numerical implementation, are space and time discretization and local interactions. Coupled map lattices (CMLs) are probably the most refined models that have been proposed for this purpose [1, 2]. The general evolution rule for the state variable at site i and time t in a 1D lattice is

$$\begin{aligned}x_i^{t+1} &= F(y_i^t) \\y_i^t &= D_\varepsilon(x_{i-1}^t, x_i^t, x_{i+1}^t) \doteq \alpha_0 x_i^t + \alpha_{-1} x_{i-1}^t + \alpha_1 x_{i+1}^t\end{aligned}\tag{1.1}$$

where $\alpha_0 = (1 - \varepsilon)$, $\alpha_{\pm 1} = \varepsilon/2$, $\varepsilon \in [0, 1]$ being the diffusion parameter. Moreover, F is some map of the interval $[0, 1]$ into itself and $x_i^t, y_i^t \in [0, 1]$.

Rule (1.1) can be interpreted both as the application of $F \circ D_\varepsilon$ to the state variables x_i^t and the application of $D_\varepsilon \circ F$ to the state variables y_i^t . Here we adopt the former interpretation and study the dynamics (1.1) on a lattice of size N with periodic boundary conditions, i.e. $x_i^t = x_{i+N}^t$.

It has been proven that in the small ε limit, the dynamics are ergodic for a special set of expanding maps [3]. Numerical simulations suggest that this is also true for a wider class of chaotic maps. On the other hand, one knows that increasing ε , new dynamical regimes appear (e.g. spatio-temporal intermittency [1, 4]). The general feature emerging from the simulations is that the ‘chaoticity’ of local dynamics is to some extent suppressed by the diffusive coupling.

Lyapunov characteristic exponents are the most direct indicators of chaotic behaviour. They can be computed by evolving the linear dynamics of vectors ξ_N^t in the tangent space of (1.1)

$$\xi_N^{t+1} = \mathbf{A}^t \xi_N^t \tag{1.2}$$

where the only non-zero elements of the $N \times N$ matrix \mathbf{A}^t are

$$(\mathbf{A}^t)_{ij} = \frac{\partial F(y_j^t)}{\partial x_j^t} = \frac{\partial F}{\partial y_j^t} \alpha_{j-i} \quad j = i - 1, i, i + 1 \tag{1.3}$$

and periodic boundary conditions imply that $(\mathbf{A}^t)_{1,0} \doteq (\mathbf{A}^t)_{1,N}$, $(\mathbf{A}^t)_{N,N+1} \doteq (\mathbf{A}^t)_{N,1}$. In particular, the maximal Lyapunov exponent Λ is defined, in the thermodynamic limit, as

$$\Lambda = \lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \ln \frac{\|\xi_N^{t+1}\|}{\|\xi_N^t\|}. \tag{1.4}$$

Apart from special cases [5], Λ cannot be exactly determined by analytical techniques. In fact, there are two main sources of difficulties. The first one originates from space-time correlations typical of any deterministic evolution rule, like (1.1). The other one follows from the non-commutative nature of process (1.2).

In order to obtain at least some approximate analytical solution, we replace the coefficients $\partial F/\partial y_j^t$ in (1.3) by a δ -correlated random process a_j^t . This amounts to a suitable random matrix approximation of the dynamics defined in (1.2) and (1.3) [6]. In formulae

$$\xi_N^{t+1} = \mathbf{R}^t \xi_N^t \tag{1.5}$$

with $(\mathbf{R}^t)_{ij} = a_j^t \alpha_{j-i}$ ($j = i - 1, i, i + 1$). Periodic boundary conditions imply for \mathbf{R}^t the same relations as for \mathbf{A}^t . We expect that this approximation works increasingly better for $\varepsilon \rightarrow 0$ and for sufficiently chaotic maps F , as these conditions guarantee small space and time correlations, respectively.

The advantage of a random matrix approach is that one can apply mean-field techniques for evaluating Λ in close analogy with those used to solve analytically the problem of directed polymers in a random medium [7]. We further restrict our investigation to the case of positive random numbers a_j^t so as to simplify the analysis (this corresponds to considering the class of maps F with an everywhere positive derivative). More precisely, the REM (random energy model) formalism [8], yielding the exact solution for sparse random matrices [7], is adapted in section 2 to Jacobi random matrices. This allows us to predict a logarithmic behaviour for Λ in the limit $\varepsilon \rightarrow 0$

$$\Lambda \sim \Lambda_0 + |\log \varepsilon|^{-1} \tag{1.6}$$

where Λ_0 is the Lyapunov exponent for $\varepsilon = 0$. Moreover, this mean-field method indicates the existence of a critical value ε_c , separating two different 'phases'.

Numerical simulations for products of Jacobi random matrices (1.5) confirm the logarithmic scaling behaviour (1.6). On the other hand, no indication of a phase transition is found. In addition simulations on a 1D CML model of expanding piecewise linear maps with positive slopes confirm both the existence of the predicted scaling behaviour, and the absence of a phase transition.

In section 3 we follow a more refined approach to the problem, based on the so-called tree approximation [9]. The results coincide with those derived in section 2. One might expect that higher-order tree approximations may remove the phase transition.

Section 4 is devoted to the analysis of the simplest version of our model, i.e. product of 2×2 random matrices. A logarithmic scaling law of Λ for $\varepsilon \rightarrow 0$ is again found by a fully analytical treatment. This allows us to explain this 'universal' property reducing the problem to the study of a suitable diffusive process.

Conclusions and perspectives are reported in section 5.

2. Mean-field approach

In this section we develop a mean-field approach to the estimate of the maximal Lyapunov exponent Λ for the product of random matrices defined in (1.5). This method is inspired by the one used in the approximate solution of the problem of directed polymers in a random medium [7].

According to definition (1.5) the generic element ξ_i^t of the Lyapunov vector is the sum of the multipliers M associated to all paths belonging to the light-cone of site (i, t) in the spacetime lattice. More precisely let us assign a factor $\varepsilon/2$ to diagonal (D) links and a factor $(1 - \varepsilon)$ to horizontal (H) links joining two neighbouring sites (see figure 1).

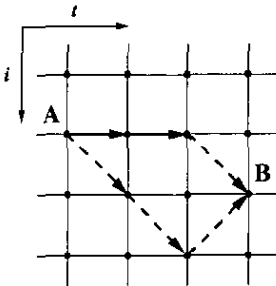


Figure 1. Some typical paths joining point A with point B through horizontal (full) and diagonal (broken) links.

The multiplier $M_p(m, T)$ associated with a generic path p of length T m with D-links can be written as

$$M_p(m, T) = \left(\frac{\varepsilon}{2}\right)^m (1 - \varepsilon)^{T-m} \prod_{k \in p} a_k \tag{2.1}$$

where a_k is a shorthand notation for $a_{i(k)}^{i(k)}$, k being a parametrization of path p . Since all components ξ_i^t are equivalent, the maximal Lyapunov exponent is given by

$$\Lambda = \lim_{T \rightarrow \infty} \frac{1}{T} \log \sum_{p,m} M_p(m, T). \tag{2.2}$$

Observe that in this approach the limit $T \rightarrow \infty$ implies the limit $N \rightarrow \infty$. It is useful to define the rate

$$\tilde{\lambda} \equiv \frac{\sum_{k=1}^T \log a_k}{T} \tag{2.3}$$

where, for the sake of simplicity, we have dropped the dependence on the path p . The probability density \tilde{P} of $\tilde{\lambda}$ satisfies the scaling law

$$\tilde{P}(\tilde{\lambda}, T) \simeq e^{-f(\tilde{\lambda})T}. \tag{2.4}$$

It is convenient to introduce the Legendre transform $L(q)$ of $f(\tilde{\lambda})$

$$f(\tilde{\lambda}) = q\tilde{\lambda} - L(q) \quad f'(\tilde{\lambda}) = q \tag{2.5}$$

where the prime denotes the derivative with respect to $\tilde{\lambda}$. If the a_k values are δ -correlated, then

$$L(q) \equiv \log \int Q(a) a^q da \tag{2.6}$$

where $Q(a)$ denotes the probability distribution of a values.

By inserting equation (2.3) into (2.1), we obtain an expression for the growth rate λ along the path p ,

$$\lambda = \frac{\log M_p(m, T)}{T} = \tilde{\lambda} + x \log \frac{\epsilon}{2} + (1-x) \log(1-\epsilon) \tag{2.7}$$

where $x \equiv m/T$ is the fraction of D-links. The probability $P(\lambda, T, x)$ to find a path of length T characterized by a growth rate λ and by a fraction x of D-links $P(\lambda, T, x)$ is obtained by substituting equation (2.7) into (2.4)

$$P(\lambda, T, x) = \tilde{P}(\lambda - (1-x) \log(1-\epsilon) - x \log(\epsilon/2), T). \tag{2.8}$$

It is worth to observing that, although the random variables a_i^t are δ -correlated, the multipliers $M_p(m, T)$ are not independent from one another because of the superposition of many different paths entering (2.2). Since a full account of these correlations is not feasible, we estimate Λ under the approximation that all multiplicative processes are reciprocally independent. This is essentially the idea of the REM introduced by Derrida [8]; the main difference in our case is that path-dependent weight factors α_i have to be assigned to the random entries of matrix (1.5).

Under this approximation the typical number $\mathcal{N}(\lambda, T, x)$ of paths of length T with a fraction x of D-links and characterized by the growth rate λ factorizes into the product of the probability in formula (2.8) by the multiplicity of such paths. Taking

into account that from each site two paths depart with a D-link and only one with an H-link, we immediately obtain

$$\mathcal{N}(\lambda, T, x) = P(\lambda, T, x) \frac{2^{Tx}(T!)}{(Tx)!(T(1-x))!}. \quad (2.9)$$

By replacing $\sum_{p,m}$ with $\int dx d\lambda \mathcal{N}(\lambda, T, x)$ in (2.2), one finally obtains

$$e^{\Lambda T} = \int dx d\lambda \exp [(-f(\tilde{\lambda}) + x \log 2 - (1-x) \log(1-x) - x \log x + \lambda)T] \quad (2.10)$$

where we have also used the Stirling formula to approximate the factorials in (2.9). The integral can be computed by the standard saddle-point method when $T \rightarrow \infty$, taking into account that $\tilde{\lambda}$ is a function of both λ and x (see (2.7)). The stationarity conditions are

$$f'(\tilde{\lambda}) = 1 \quad f'(\tilde{\lambda}) \log \frac{2(1-\varepsilon)}{\varepsilon} = \log \frac{2(1-x)}{x}. \quad (2.11)$$

Hence,

$$x = \varepsilon \quad (2.12)$$

i.e. the fraction of D-links maximizing the contribution to the Lyapunov exponent coincides with the coupling strength itself. Upon substituting equations (2.7) and (2.12) into (2.10), and neglecting corrections to the saddle point, we find

$$\Lambda = \tilde{\lambda} - f(\tilde{\lambda}). \quad (2.13)$$

From the first part of (2.5) it is readily seen that Λ can be written in the more compact form

$$\Lambda = L(1) \quad (2.14)$$

where $L(1)$ is the generalized Lyapunov exponent [10] for $q = 1$ associated with the scalar multiplicative process. Notice that Λ is independent of the coupling constant ε .

This result holds only if the typical number of paths maximizing the integral in (2.10) is larger than 1, otherwise there are no paths, for $T \rightarrow \infty$, verifying the stationarity conditions (2.11) (this is a well known argument in the standard solution of the REM model [8]). In such a case, the Lyapunov exponent is determined by setting the exponential growth rate of $\mathcal{N}(\lambda, T, x)$ equal to zero,

$$f(\tilde{\lambda}) + x \log \frac{1}{2}x + (1-x) \log(1-x) = 0. \quad (2.15)$$

More precisely, once $\tilde{\lambda}$ is determined from (2.15) and the second part of (2.11) (in fact, the maximization over x values is still meaningful), the Lyapunov exponent is simply given by the corresponding λ value. Notice that in this phase the equality $f' = 1$ no longer holds, and x is different from ε .

One may ask whether both phases can indeed be observed by varying the coupling strength ε . The second one is equivalent to the low-temperature phase in the REM model and certainly holds in some neighbourhood around $\varepsilon = 0$, where clearly $\Lambda = \Lambda_0 = \lim_{q \rightarrow 0} L(q)/q$, i.e. the Lyapunov exponent coincides with the characteristic exponent of the random process a_k . The presence of the first phase depends upon the existence of a solution of the equation

$$-\varepsilon_c \log \frac{1}{2}\varepsilon_c - (1 - \varepsilon_c) \log(1 - \varepsilon_c) = f(\tilde{\lambda}). \quad (2.16)$$

Here, the function f is computed for the value of the argument where f' is equal to 1. As the maximum value of the left-hand side (achieved at $\varepsilon = \frac{2}{3}$) is $\log 3$, f must be smaller than such a value, in order to have the transition. Therefore, the first phase can be observed only for suitable probability distributions $Q(a)$.

An interesting point is related to the scaling dependence of the Lyapunov exponent on the coupling strength. To this aim, let us expand $f(\tilde{\lambda})$ around the maximum Λ_0

$$f(\tilde{\lambda}) = \beta \delta_\lambda^2 \quad (2.17)$$

where $\delta_\lambda = (\lambda - \Lambda_0)$. Let us now substitute (2.17) into (2.15) and (2.11). Taking the leading order in ε and x as both tend to zero, the latter equations reduce to

$$2\beta \delta_\lambda \log \varepsilon = \log x \quad \beta \delta_\lambda^2 = -x \log x. \quad (2.18)$$

After solving for δ_λ from the first part of (2.18) and substituting in the second one, we find an equation for x

$$\log x = -4\beta x (\log \varepsilon)^2. \quad (2.19)$$

By introducing the auxiliary variables $u = -\log x$ and $\phi = 4\beta (\log \varepsilon)^2$, (2.19) reads as

$$\log u = \log \phi - u. \quad (2.20)$$

Since, for $\varepsilon \rightarrow 0$, i.e. $\phi \rightarrow \infty$, the leading-order solution of (2.20) is $u = \log \phi$, one easily obtains

$$x = \frac{\log |\log \varepsilon| + \mathcal{O}(\log \log |\log \varepsilon|)}{2\beta (\log \varepsilon)^2}. \quad (2.21)$$

Thus we see that x goes to zero, although more slowly than ε . By substituting in the first part of (2.18) we find the solution for δ_λ ,

$$\delta_\lambda \simeq -\frac{\log |\log \varepsilon|}{\beta \log \varepsilon}. \quad (2.22)$$

The expression for Λ is obtained from (2.7) neglecting next-to-leading order terms, both in x and ε

$$\Lambda = \Lambda_0 + \delta_\lambda + x \log \varepsilon. \quad (2.23)$$

Inserting equations (2.21) and (2.22) into (2.23) yields,

$$\Lambda = \Lambda_0 - \frac{\log |\log \varepsilon|}{2\beta \log \varepsilon} \tag{2.24}$$

which indicates a very slow convergence towards the limit value Λ_0 for $\varepsilon = 0$.

We cannot expect that (2.24) holds exactly, as the model equations have been derived disregarding all kinds of correlation. Nevertheless, this approximation is sufficient to catch the relevant features.

We have performed numerical simulations with products of random matrices of various sizes, and uniform distributions $Q(a)$ over different intervals $[a_{\min}, a_{\max}]$. In figure 2 we report

$$\Sigma \equiv (\Lambda - \Lambda_0) |\log_{10} \varepsilon| \tag{2.25}$$

against ε for 100×100 matrices (for matrices of this size, Λ turns out to be almost independent of N (see (1.4))). Σ clearly converges towards a constant value for $\varepsilon \rightarrow 0$. The constancy of Σ implies that the leading behaviour $|\log \varepsilon|^{-1}$ is confirmed, while no evidence is found of the further correction factor $\log |\log \varepsilon|$. This should not be considered as a major drawback, since (2.24) is based on a model which neglects all kinds of correlation.

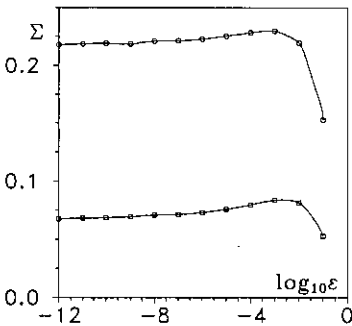


Figure 2. Σ (see equation (2.23)) against ε for $a_{\max} = 3$, $a_{\min} = 1$ (squares), $a_{\min} = \frac{1}{3}$ (circles) from products of 100×100 random matrices. The curves are guides for the eyes.

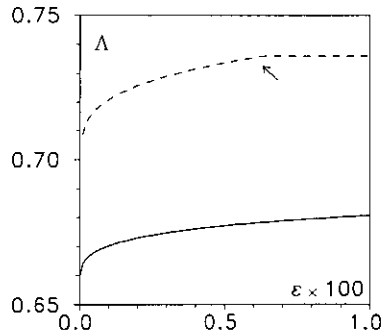


Figure 3. Λ against ε for $a_{\min} = 1$ and $a_{\max} = 3$ (continuous curve), compared with the theoretical prediction (broken curve). The arrow indicates the transition point as from the mean-field approach. Note also that the asymptotic value $\Lambda(0) = \frac{3}{2} \log 3 - 1 = 0.6479\dots$ lies under the lower edge.

In figure 3, Λ is plotted against ε for the same parameter values as in figure 2. We do not observe any phase with constant Λ and even more, no evidence of non-analytic behaviour.

Simulations have also been performed directly for a CML model. We have chosen the map F as in figure 4, such that all multipliers are positive. The results for Σ , reported in figure 5, again confirm the scaling law (1.6). This last result suggests also that this scaling law is largely independent of the probability distribution $Q(a)$. In fact, in the case of the map in figure 4, $Q(a) = \gamma \delta(a - 9) + (1 - \gamma) \delta(a - 9/4)$, $\gamma \in [0, 1]$, substantially different from the flat distributions used in the simulations with

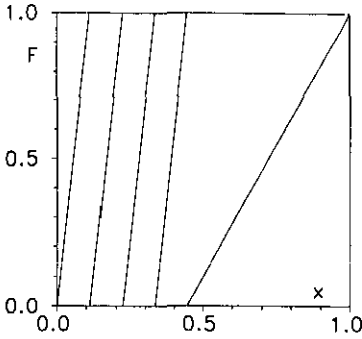


Figure 4. The local map $F(x)$ chosen for numerical simulations. The two slopes are 9 and $9/5$.

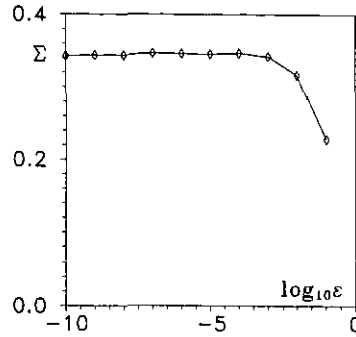


Figure 5. Σ against ϵ for a chain of 100 maps as in figure 4.

the random matrices. Let us also observe that in the CML case the distribution $Q(a)$ does depend on ϵ through the weight γ . Nevertheless our numerical results show that this does not affect the leading term of the scaling law.

3. The tree approximation

Let us consider the multiplicative process (1.2) from a different point of view. The elements of the vector ξ^t are updated according to the rule

$$\xi_i^{t+1} = a((1 - \epsilon)\xi_i^t + \frac{1}{2}\epsilon(\xi_{i-1}^t + \xi_{i+1}^t)) \tag{3.1}$$

where a is the random multiplier.

Following the dictionary of directed polymers, a is parametrized as $\exp(-\beta V)$, where V is a random potential with probability distribution $\rho(V)$, and β is the inverse temperature. The problem is therefore reshaped as follows

$$\xi_i^{t+1} = \exp^{-\beta(V - \ln(1-\epsilon))} \xi_i^t + \exp^{-\beta(V - \ln(\epsilon/2))} (\xi_{i-1}^t + \xi_{i+1}^t) \tag{3.2}$$

where the multiplicative factors $(1 - \epsilon)$ and $\epsilon/2$ are raised to the power β . It is worth observing that β here is a fictitious parameter which will be set equal to 1 in the end.

The approximation consists in neglecting correlations after one step in the recursive process. More precisely, the following evolution equation for the probability distribution $P(\xi)$ of ξ is written

$$P_{t+1}(\xi) = \int d\xi_{i-1}^t d\xi_i^t d\xi_{i+1}^t dV \rho(V) P_t(\xi_{i-1}) P_t(\xi_i) P_t(\xi_{i+1}) \delta(\xi - \xi_i^{t+1}). \tag{3.3}$$

This is called tree approximation in the theory of directed polymers [9]. Some informations on the solution of (3.3) are obtained by introducing the generating function

$$G_t(x) \equiv \langle \exp(-\exp^{-\beta x} \xi) \rangle_t \tag{3.4}$$

where the average $\langle \cdot \rangle_t$ is performed over the probability distribution $P_t(\xi)$. According to (3.3), $G_t(x)$ evolves as

$$G_{t+1}(x) = \int \rho(V) dV G_t(x + V - \ln(1 - \epsilon)) G_t^2(x + V - \ln(\epsilon/2)). \tag{3.5}$$

The initial condition is given by

$$G_0(x) = \exp(-\exp^{-\beta x}). \tag{3.6}$$

The generating function has the following property

$$\lim_{x \rightarrow -\infty} G_t(x) = 1 \quad \lim_{x \rightarrow -\infty} G_t(x) = 0. \tag{3.7}$$

The value $G_t = 1$ is an unstable fixed point of the map (3.5), while $G_t = 0$ is a stable fixed point. In analogy with directed polymers, the dependence on β is entirely contained in the initial conditions. In our case this is a straightforward consequence of the initial choice of raising to the power β the ϵ -dependent factors in (3.2).

Equation (3.5) admits travelling wave solutions [9]

$$G_t(x) = w(x - ct). \tag{3.8}$$

The asymptotic behaviour of $w(y)$ is, from equations (3.4), (3.6),

$$w(y) \sim 1 - \exp^{-\beta y}. \tag{3.9}$$

This equation shows that the product $c\beta$ corresponds with the usual definition of the Lyapunov exponent.

By substituting equations (3.8) and (3.9) into (3.5) and neglecting higher-order terms, one obtains the following equation for the wave velocity

$$c(\beta, \epsilon) = \begin{cases} 1/\beta \ln[\int \rho(V) dV \exp(-\beta V)((1 - \epsilon)^\beta + 2(\epsilon/2)^\beta)] & \text{if } \beta < \beta_c(\epsilon) \\ c(\beta_c, \epsilon) & \text{if } \beta > \beta_c(\epsilon) \end{cases} \tag{3.10}$$

where $\beta_c(\epsilon)$ is implicitly defined by

$$\left. \frac{dc(\beta, \epsilon)}{d\beta} \right|_{\beta=\beta_c} = 0. \tag{3.11}$$

This relation represents a critical line in the parameter space (β, ϵ) . Its intersection with the line $\beta = 1$ yields the critical value ϵ_c

$$\ln \left((1 - \epsilon_c)^{1-\epsilon_c} \left(\frac{\epsilon_c}{2} \right)^{\epsilon_c} \right) = \ln \langle a \rangle - \frac{\langle a \ln a \rangle}{\langle a \rangle} \tag{3.12}$$

where the averages are performed over the given probability distribution of a . One can notice that in the derivation of (3.12) it has been crucial to begin with a generic value of β .

The critical value ϵ_c defined by (3.12) coincides with the one obtained with the mean-field method in section 2. In fact, one has to impose that equations (2.10) and (2.14) hold simultaneously. Then, from the Legendre transform (2.4), observing that $\bar{\lambda} = L'(q)$, one obtains

$$-f(\bar{\lambda}) = L(1) - L'(1) \tag{3.13}$$

which manifestly coincides with (3.12).

Finally, the Lyapunov exponent is given by

$$\begin{aligned} \Lambda &= \beta_c c(\beta_c, \epsilon) && \text{if } \epsilon < \epsilon_c \\ \Lambda &= c(1, \epsilon) && \text{if } \epsilon > \epsilon_c. \end{aligned} \tag{3.14}$$

This last result holds under the assumption that

$$\begin{aligned} \beta_c(\epsilon) < 1 &&& \text{if } \epsilon < \epsilon_c \\ \beta_c(\epsilon) > 1 &&& \text{if } \epsilon > \epsilon_c \end{aligned} \tag{3.15}$$

which has been numerically verified for some probability distributions $Q(a)$. In fact, from (3.10), one can use the fact that $c(\beta, \epsilon)$ is constant for a given value of ϵ if $\beta > \beta_c$.

From (3.14) and (3.10), one obtains the same expressions for the maximal Lyapunov exponent as those derived in the previous section with the mean-field method.

4. Two-dimensional model

In order to better comprehend the origin of the scaling behaviour of the maximal Lyapunov exponent, we now study the simplest non-trivial case: 2×2 random matrices of the form (1.5). The evolution of the two components u and v of a vector ξ is described by the recursive relation

$$\begin{aligned} u_{n+1} &= a_n((1 - \epsilon)u_n + \epsilon v_n) \\ v_{n+1} &= b_n(\epsilon u_n + (1 - \epsilon)v_n) \end{aligned} \tag{4.1}$$

where a_n and b_n represent two independent realizations of the same random process, with $a_n, b_n > 0$.

Equation (4.1) can be written as a recursive equation for u_n only,

$$u_{n+1} = (1 - \epsilon)a_n \left(1 + \frac{b_{n-1}}{a_{n-1}}\right) u_n + a_n b_{n-1} (\epsilon^2 - (1 - \epsilon)^2) u_{n-1}. \tag{4.2}$$

This equation can be transformed into a 1D nonlinear equation by introducing

$$S_n \equiv u_n / a_{n-1} u_{n-1}. \tag{4.3}$$

We obtain

$$S_{n+1} = (1 - \epsilon) \left(1 + \frac{b_{n-1}}{a_{n-1}}\right) - (1 - 2\epsilon) \frac{b_{n-1}}{a_{n-1}} \frac{1}{S_n}. \tag{4.4}$$

Notice that S_n represents the multiplicative correction to the uncoupled (i.e. $\epsilon = 0$) process, so that it is a proper variable to estimate the variation of the Lyapunov exponent.

From (4.4) we also notice that the two random processes always occur via their ratio

$$c_n \equiv b_{n-1}/a_{n-1}. \tag{4.5}$$

Therefore, (4.4) can be rewritten as

$$S_{n+1} = (1 - \epsilon)(1 + c_n) - (1 - 2\epsilon)c_n/S_n. \tag{4.6}$$

For $\epsilon = 0$, $S_n = 1$ is a fixed point of (4.5), in agreement with the interpretation of the variable S_n . If one is interested in the small ϵ limit, it is useful to introduce the new variable

$$R_n = S_n - 1. \tag{4.7}$$

Equation (4.5) can be finally written as

$$R_{n+1} = \frac{c_n R_n}{1 + R_n} - \epsilon \left(1 + c_n - \frac{2c_n}{1 + R_n} \right). \tag{4.8}$$

The structure of (4.8) reminds us an analogous equation derived to describe anomalous diffusion in a 1D chain characterized by random potentials [11]. In that case the starting Master equation was mapped onto a Schrödinger problem and then transformed into a transfer matrix equation for the spectrum. The multiplicative noise was shown to lead to a new type of intermittency phenomenon with entirely different scaling laws. Here, we show that a similar scaling law is found, despite the fact that it is not explained in terms of some intermittent mechanism.

Let us first investigate the case $\epsilon = 0$. The hyperbolic map (4.8) crosses the bisectrix in $R^{(0)}(c_n) = 0$ and $R^{(1)}(c_n) = c_n - 1$. For $c_n > 1$ ($c_n < 1$) the slope $\sigma_n = \partial R_{n+1}/\partial R_n$ is larger (smaller) than 1 in $R^{(0)}$ and smaller (larger) than 1 in $R^{(1)}$ (observe that for $c_n = 1$, $R^{(0)} = R^{(1)}$). Being $R^{(0)}$ independent of the random variable c_n , it is a true fixed point of (4.8). In the linear approximation around $R^{(0)}$, $R_n = \prod_{i=0}^{n-1} c_i R_0$. Since c_i is equally probable to $(c_i)^{-1}$, the fixed point $R^{(0)}$ is marginally stable. As a consequence, its stability is determined by the nonlinear contribution: for $R_0 > 0$, it is straightforward to show that $R_n/R_0 < \prod_{i=0}^{n-1} c_i$, i.e. $R^{(0)}$ is nonlinearly stable from the right. The same argument shows that $R^{(0)}$ is unstable from the left.

For small ϵ and $|\delta_n| > \epsilon$ (with $\delta_n = 1 - c_n$) the two intersections of (4.8) with the bisectrix at order ϵ^2 are

$$R^{(0)}(c_n) = -\epsilon + \frac{\epsilon^2 c_n}{1 - c_n} \quad R^{(1)}(c_n) = c_n - 1 - \epsilon c_n - \frac{\epsilon^2 c_n}{1 - c_n} \tag{4.9}$$

where the dependence of both $R^{(0)}$ and $R^{(1)}$ on ϵ is understood. Note that $R^{(0)}$ now does depend on c_n at order ϵ^2 . When $|\delta_n| < \epsilon$, i.e. when the hyperbolic map (4.8) is nearly tangent to the bisectrix in the origin, the two intersections are, at order ϵ ,

$$R^{(\pm)} = \frac{\delta_n}{2} - \epsilon \pm \sqrt{\epsilon^2 + \frac{\delta_n^2}{4}}. \tag{4.10}$$

It is important to observe that, since the term in the square root of (4.10) is positive definite, the two intersections always exist also in this case.

We can now show that the intersection $R^{(u)}(c_n)$ with slope σ_n larger than 1 is smaller than $-\epsilon$ for all c_n , while the intersection $R^{(s)}(c_n)$ with slope σ_n smaller than 1 is larger than $-\epsilon$ for all c_n . In fact, from equations (4.9) and (4.10) it follows that

$$\begin{aligned}
 R^{(u)}(c_n) &= R^{(0)}(c_n) & R^{(s)}(c_n) &= R^{(1)}(c_n) & \text{if } c_n > 1 + \epsilon \\
 R^{(u)}(c_n) &= R^{(-)}(c_n) & R^{(s)}(c_n) &= R^{(+)}(c_n) & \text{if } 1 - \epsilon < c_n < 1 + \epsilon \\
 R^{(u)}(c_n) &= R^{(1)}(c_n) & R^{(s)}(c_n) &= R^{(0)}(c_n) & \text{if } c_n < 1 - \epsilon.
 \end{aligned}
 \tag{4.11}$$

One can immediately notice that $R^{(u)}(c_n) < -\epsilon - \epsilon^2 = A$, and $R^{(s)}(c_n) > -\epsilon + \epsilon^2 = B$. Now we can prove that, if $R_0 > B$, then $R_n > B \forall n$. In fact, from $B > A$ it follows that $R_0 > R^{(u)}(c_0)$ for all c_0 . The only c_0 values giving $R_1 < R_0$ are those for which $R^{(s)}(c_0) < R_0$, but then $R_1 > B$. This concludes the proof.

Accordingly, for large values of R_n , nonlinear terms prevent R_n from diverging to $+\infty$, while for $R_n \simeq B$, ϵ^2 terms prevent $R_n + \epsilon$ from becoming smaller than ϵ^2 . In particular, this last mechanism excludes the existence of any intermittent phenomenon. Therefore, the dynamics of R_n is bounded both from above and below. Within these bounds, and for ϵ small enough, we can linearize the equations of motion around $-\epsilon$ and neglect ϵ^2 terms. We find that $w_n \equiv \log(R_n + \epsilon)$ exhibits a diffusive motion

$$w_{n+1} = w_n + \log c_n. \tag{4.12}$$

Accordingly, the dynamical behaviour can be modelled as a pure diffusive process in the interval $w_{\min} = 2 \log \epsilon + g_1 < w_n < g_2 = w_{\max}$, with g_1 and g_2 independent of ϵ . Moreover, the previously discussed confinement mechanisms of the dynamics can be schematized as reflecting barriers set at w_{\min} and w_{\max} . Hence, the probability $P(w)dw$ to find w between w and $w + dw$ is expected to be flat inside $[w_{\min}, w_{\max}]$ except for some deviations in two finite regions around the extrema of this interval. This conjecture has been numerically verified (see figure 6). We will see that the unknown constants g_1 and g_2 contribute only to the prefactors, while they do not affect the scaling law of the Lyapunov exponent.

Let us now link the probability density $P(w)$ with the Lyapunov exponent

$$\Lambda = \frac{1}{T} \left(\sum_n \log a_n + \sum_n \log S_n \right). \tag{4.13}$$

The first term in the right-hand side of (4.13) is the Lyapunov exponent Λ_0 of the uncoupled case. Recalling (4.7) we obtain

$$\Lambda \simeq \Lambda_0 + \frac{\sum_n R_n}{T} = \Lambda_0 + \langle R \rangle \tag{4.14}$$

where only the first-order term in R_n has been retained. The average R value can be determined from the w distribution

$$\langle R \rangle \simeq \int_{2 \log \epsilon + g_1}^{g_2} \frac{e^w - \epsilon}{g_2 - g_1 - 2 \log \epsilon} dw. \tag{4.15}$$

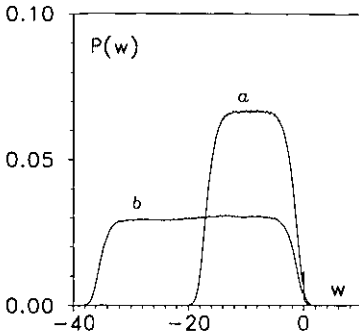


Figure 6. Probability distribution of w for 2×2 random matrices; a_n and b_n are randomly chosen from a uniform distribution in the interval $[1/3, 3]$. Curves a and b refer to $\epsilon = 10^{-4}$ and $\epsilon = 10^{-6}$, respectively.

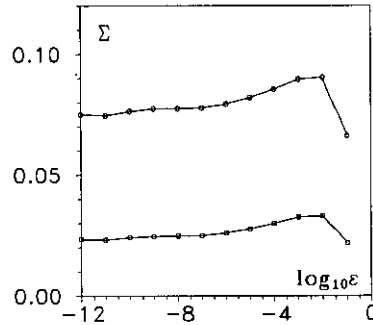


Figure 7. Σ against ϵ for 2×2 random matrices; circles and dots refer to the same parameter values as those reported in figure 2 for 100 matrices.

Neglecting higher-order terms in ϵ , we obtain

$$\langle R \rangle \simeq -\frac{e^{g_2}}{2 \log \epsilon}. \tag{4.16}$$

Such a scaling behaviour coincides with the one found in the previously mentioned Schrödinger problem [11], although we are not in the presence of any intermittency. Notice also that only g_2 contributes to the leading term. This is somehow clear because only large R_n values give relevant contributions to the average.

We have performed numerical simulations for the same random processes studied in section 2. The results are shown in figure 7. The logarithmic behaviour is again perfectly confirmed. We also see that the asymptotic Σ value is significantly smaller than in the high-dimensional case.

5. Conclusions and perspectives

We have studied the scaling law of the maximal Lyapunov exponent Λ when $\epsilon \rightarrow 0$ for coupled map lattices, where the local map F is chaotic and for products of Jacobi random matrices, which can be thought of as an approximation of the chaotic dynamics.

Analytical and numerical methods show the presence of a logarithmic behaviour as $\epsilon \rightarrow 0$:

$$\Lambda \sim \Lambda_0 + |\log \epsilon|^{-1}.$$

This scaling law is derived by an extension of the techniques used for solving the REM [8], which amounts to neglect correlations among paths. The introduction of a partial correlation by a proper tree approximation [9] does not modify this scaling behaviour. Both analytical treatments suggest the presence of a phase transition at ϵ_c , which is not seen in numerical simulations. Higher-order tree approximations [12] might remove this phase transition.

Λ turns out to be an increasing function of ε . This result seemingly contradicts the statement that the diffusive coupling tends to decrease the 'chaoticity' of a CML. In fact, we expect that a more appropriate global indicator as the Kolmogorov–Sinai entropy (i.e. the sum of all the positive Lyapunov exponents) should decrease for increasing ε .

In the special case of 2×2 random matrices we have identified the dynamical mechanism which leads to the scaling law. It is a diffusive process of a proper variable inside a finite interval, whose lower limit depends on ε . One could conjecture that the same mechanism is present also in matrices of higher rank, although the mathematical treatment cannot be a direct extension of the 2×2 case.

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